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Asymptotic theory of an oscillating wing section in weak ground effect

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Abstract

This study addresses unsteady aerodynamic forces acting on a wing section oscillating in a steady incompressible (and inviscid) uniform flow in presence of a distant flat ground. Three fundamental dimensionless parameters characterize the magnitude of those forces: the ratio δ of the wing transversal displacement to its chord, the ratio ε of the wing chord to its average distance from the ground, and the ratio k of the wing chord to the distance traveled by the flow during one oscillation period. With the first two serving as small parameters, asymptotic series of the form $\delta f_0(k) + \delta\varepsilon^2 f_1(k) + \delta\varepsilon^2 g(k/\varepsilon) f_2(k) + \cdots$ have been constructed for the wing lift and pitching moment. In the case of heave oscillations, three-terms-series for the lift fits nicely the available numerical data for wide range of δ, ε and k values.

Keywords: Unsteady aerodynamics; Ground effect; Asymptotic solution

1. Introduction

The problem of a thin wing section executing small-amplitude oscillations in a steady incompressible inviscid uniform flow has been thoroughly studied in the past. In those cases where the flow is unbounded, there are several theories that can provide simple closed-form analytical expressions relating aerodynamic loads on the wing with the characteristics of the wing motion. Examples can be found practically in any reference on aeroelasticity, e.g., in Bisplinghoff et al. [1].

In those cases where the flow is bounded by an impermeable plane (as a ground), the derivations involved in obtaining the aerodynamic loads become algebraically complicated. In fact, no closed-form analytical expressions are known for the loads when the distance between the wing and the bounding plane can take any value in $(0, \infty)$. Yet, when this distance can be assumed either small or large as compared with the wing's (quarter) chord, it becomes possible to obtain the loads in the form of asymptotic series. A recent compilation of the theories addressing the former case (where the distance is small) can be found in the book of Rozhdestvensky [2].

We have found only a single account of an asymptotic theory addressing the latter case (where the distance is large). This theory was developed by Panchenkov [3] to obtain hydrodynamic loads acting on a hydrofoil oscillating

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below the water surface. Being concerned with a free boundary (representing the water surface), this problem is obviously much more complicated than the one concerned with a fixed boundary, and it includes the latter as its sub-case. An appropriate limiting procedure can reduce (degenerate) Panchenkov results so as to apply to the case of a fixed planar boundary, but derivations involved in this procedure turn out to be more complicated than those required to solve the fixed-boundary problem directly. Moreover, it will be shown in Section 4 that Panchenkov's series for aerodynamic loads suffer from impractically small convergence radius. All this comes to imply that there is no adequate theory addressing the (weak) effect of a distant fixed planar boundary on unsteady aerodynamic forces acting on an oscillating wing; this effect is the subject matter of the present exposition.

The problem of an oscillating wing section in (weak) ground effect has four fundamental length scales from which three independent dimensionless ratios can be constructed. One of them is the ratio δ of the transversal displacement of the wing to the wing chord, the other is the ratio ε of the wing (quarter) chord to its average distance from the ground, and the third is the ratio k of the wing (semi) chord to the distance traveled by the wing during one oscillation period. As δ goes to zero, the wing ceases being a perturbation to the flow; as k goes to zero, the problem reduces to that of a non-oscillating wing section in ground effect; as ε goes to zero, the problem reduces to that of an oscillating wing section out of ground effect.

A posteriori we can identify two major obstacles in the way of obtaining a practically useful asymptotic solution of this apparently regular perturbation problem. The first one is related with inherent non-linearity of the ground effect with respect to δ. In fact, the first five terms in the asymptotic series for the lift of a non-oscillating wing ($k = 0$) involve $\delta, \delta^2, \delta^3, \delta\varepsilon^2$ and $\delta^2\varepsilon$ – see, for example, in Coulliette and Plotkin [4]. In the case of an oscillating wing all terms which are non-linear in δ present a major complication since they do not allow frequency separation. We chose to avoid this obstacle, and hence the present study is confined only to the terms which are linear in δ; it is tacitly assumed that there exist a sufficiently small δ that renders this linear approximation valid. Pertinent linearization procedure can be found in Sections 2 and 3.

The second obstacle is related to the fact that the equation governing the linear-in-δ part of the aerodynamic loads depends not only on ε and k, but also on the ratio k/ε of the two. This creates an unusual situation where textbook-standard approach fails to provide a practically useful asymptotic solution of this equation with respect to ε. If k is assumed an independent parameter, the terms involving k/ε alone make the solution suffer from an impractically small radius of convergence; if k/ε is assumed as an independent parameter, the terms involving k alone make the solution suffer from slow convergence rate. Sections 4 and 5 are devoted to the description of this problem and its ultimate solution.

2. Formulation

Consider an infinite domain of incompressible fluid of density ρ bounded by an impermeable planar wall (representing a solid ground) and a thin wing of chord $2b$ and infinite span moving with constant average velocity U parallel to that wall and oscillating in due course. b and U, as well as their combinations, $b/U, bU, 1/2\rho U^2, \rho U^2 b$ and $2\rho U^2 b^2$ will serve as convenient units of length, velocity, time, velocity potential, pressure, force per unit span and moment per unit span, respectively.

The fluid will be assumed inviscid throughout, and hence the existence of the thin vortical wake starting at the trailing edge and extending to infinity has to be postulated. The solid or vortical, and, in any case, non-potential interior of the wing and its wake will be treated as a single entity; the flow will be assumed irrotational in its exterior.

A right handed, Cartesian reference frame, following the wing along the ground, is, perhaps, the most appropriate for the case considered. For the sake of definiteness, its x-axis will be assumed pointing backward, along the flow direction relative to the wing and leveled with the ground plane, whereas its z-axis will be assumed pointing upward through the mid-chord. Relative to this system, the upper (the farther from the ground, marked by '$+$') and the lower (the closer to the ground, marked by '$-$') surfaces of the wing-wake entity can be defined by

$$z = h + \delta f_\pm(t, x), \tag{1}$$

with x spanning the interval $[-1, \infty)$, and t (time) taking on any value in the interval $(-\infty, \infty)$; the last implication will be tacitly understood hereafter. Here, h is the average distance between the wing's trailing edge and the ground; $h \pm \delta$ are bounds of the instantaneous maximum and minimum distance between the upper and the lower surfaces and the ground; and f_+ and f_- are the shape functions of the respective surfaces.

It will be assumed that δ is small as compared with unity, whereas f_+, f_- and their first partial derivatives are of the order of unity (implying that the transition between the wing and the wake is smooth). Accordingly, if the thickness of the wing-wake does not vary with time, the functions f_+ and f_- describing its upper and lower surfaces can be approximated as superposition

$$f_\pm(t,x) = \bar{f}_\pm(x) + f(t,x) + \mathrm{O}(\delta^2) \tag{2}$$

of time independent functions f_+ and f_- describing its thickness and camber distributions, and a single function f, describing the shape of its motion.

With the flow assumed irrotational in the exterior of the wing-wake entity, one can always define scalar velocity potential, and hence bring the problem of finding aerodynamic loads acting on the wing to that of finding a harmonic function satisfying pertinent boundary conditions. Of these, the impermeability condition at the wing and free-surface condition at the wake are non-linear, and hence the next step involves linearization with respect to δ. It can be carried out in several ways, but as long as h and δ are independent, the procedure outlined in Ashley and Landahl [5] can be used 'as is'. At its outcome, the problem reduces to that of finding a harmonic function ϕ (the leading term in the expansion of the outer perturbation potential with respect to δ) on the upper-half-plane-exterior of a slit $\{(x,z)\colon x \in [1,\infty) \text{ and } z = h\}$ (the leading order representation of the interior of the wing-wake) that vanishes at infinity, and satisfies

$$\frac{\partial}{\partial z}\phi(t,x,z) = 0 \quad \text{for each } x \in (-\infty,\infty) \text{ and } z = +0; \tag{3}$$

$$\frac{\partial}{\partial z}\phi(t,x,z) = \delta\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\bar{f}_\pm(x) + f(t,x)\right) \quad \text{for each } x \in (-1,1) \text{ and } z = h \pm 0; \tag{4}$$

$$p(t,x) = 0 \quad \text{for each } x \in [1,\infty). \tag{5}$$

Here, p is (the leading order term in the expansion of) the pressure jump across the slit, related with the respective potential jump,

$$\mu(t,x) = \phi(t,x,h+0) - \phi(t,x,h-0), \tag{6}$$

by the variant of the Bernoulli's theorem,

$$p(t,x) = -2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\mu(t,x). \tag{7}$$

The most appealing feature of this leading-order formulation is the possibility to separate oscillatory and time-averaged aerodynamic loads acting on the wing. In fact, upon removing the time-averaged constituents of Eqs. (3)–(7) and resetting ϕ, μ and p to represent hereafter the unsteady (zero-mean) constituents of the respective quantities, one will find that these new ϕ, μ and p fit exactly the same Eqs. (3), (5), (6) and (7) – but instead of (4),

$$\frac{\partial}{\partial z}\phi(t,x,z) = \delta\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)f(t,x) \quad \text{for each } x \in (-1,1) \text{ and } z = h \pm 0. \tag{8}$$

Identical for the two sides of the slit, this equation can be interpreted as an impermeability condition for an infinitesimally thin wing which motion is described by f. In other words, in the leading-order approximation with respect δ, the problem of finding unsteady constituents of aerodynamic loads a thin an oscillating wing is equivalent with that of finding aerodynamic loads on an infinitesimally thin oscillating wing with a rectilinear infinitesimally thin wake behind it.

3. Integral equation for μ

A common technique of finding the flow field bounded by an impermeable planar wall is by extending it over the entire infinite domain and imposing symmetry with respect to the plane that previously was the wall. In the present case it implies extending the flow field over the entire x–z plane and placing an inverted (image) wing-wake at the distance h 'below' the ground, at $z = -h$.

By the second Green theorem (or, rather, by the Biot–Savart law directly) the velocity at any point of the flow field can be expressed in terms of the of the potential jump μ; in particular, the z-velocity-component can be written as

$$\frac{\partial\phi(t,x,z)}{\partial z} = -L\{\mu;t,x,z-h\} + L\{\mu;t,x,z+h\}, \tag{9}$$

where L stands for the operator

$$L\{\mu;t,x,z\} = \frac{1}{2\pi}\int_{-1}^{\infty}\frac{\partial\mu(t,x')}{\partial x'}\frac{(x-x')\,\mathrm{d}x'}{(x-x')^2+z^2}; \tag{10}$$

pertinent formulae can be found practically in any reference on aerodynamics, for example, in Ashley and Landahl [5].

The conjunction of (8) and (9) yields an integral equation,

$$L\{\mu;t,x,+0\} - L\{\mu;t,x,2h\} = -\delta\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)f(t,x) \quad \text{for each } x\in(-1,1), \tag{11}$$

for μ. Supplemented by the free surface condition,

$$\mu(t,x) = \mu(t-x+1,1) \quad \text{for each } x\in(1,\infty), \tag{12}$$

stemming from the conjunction of (5) and (7), and

$$\mu(t,-1) = 0, \tag{13}$$

stemming from the continuity of the velocity potential in the vicinity of the leading edge, Eq. (11) completely determines μ.[1] Our ultimate task is to solve it – once μ is found, integral aerodynamic loads immediately follow. In particular, the lift of the wing and the pitching moment about its mid-chord can be computed with

$$c_z(t) = -\frac{1}{2}\int_{-1}^{1}p(t,x)\,\mathrm{d}x = \mu(t,1) + \frac{\mathrm{d}\mu(t,1)}{\mathrm{d}t} - \int_{-1}^{1}\frac{\partial^2\mu(t,x)}{\partial x\partial t}x\,\mathrm{d}x, \tag{14}$$

$$c_m(t) = \frac{1}{4}\int_{-1}^{1}p(t,x)x\,\mathrm{d}x = -\frac{1}{4}\frac{\mathrm{d}\mu(t,1)}{\mathrm{d}t} + \frac{1}{4}\int_{-1}^{1}\left(x^2\frac{\partial^2}{\partial x\partial t} - 2x\frac{\partial}{\partial x}\right)\mu(t,x)\,\mathrm{d}x. \tag{15}$$

Since Eqs. (11)–(15) are linear, they can be reformulated for the case where the wing executes harmonic oscillations with (reduced) frequency k, rather then general periodic motion; solution for the latter will follow the former using Fourier integrals. Accordingly, we set

$$f(t,x) = \mathrm{Re}\big(\hat f(x)\mathrm{e}^{\mathrm{i}kt}\big), \qquad \mu(t,x) = \delta\,\mathrm{Re}\big(\hat\mu(x)\mathrm{e}^{\mathrm{i}kt}\big), \tag{16a}$$

$$c_z(t) = \delta\,\mathrm{Re}(\hat c_z\mathrm{e}^{\mathrm{i}kt}), \qquad c_m(t) = \delta\,\mathrm{Re}(\hat c_m\mathrm{e}^{\mathrm{i}kt}), \tag{16b}$$

with Eqs. (11)–(15) taking on the respective forms,

$$\hat L_k\{\hat\mu;x,+0\} - \hat L_k\{\hat\mu;x,2h\} = \hat G_0(x) \quad \text{for each } x\in(-1,1), \tag{17}$$

$$\hat\mu(x) = \hat\mu(1)\mathrm{e}^{-\mathrm{i}k(x-1)} \quad \text{for each } x\in(-1,\infty), \tag{18}$$

$$\hat\mu(-1) = 0, \tag{19}$$

$$\hat c_z = (1+\mathrm{i}k)\hat\mu(1) - \mathrm{i}k\int_{-1}^{1}\frac{\mathrm{d}\hat\mu(x)}{\mathrm{d}x}x\,\mathrm{d}x, \tag{20}$$

$$\hat c_m = -\frac{\mathrm{i}k}{4}\hat\mu(1) + \frac{1}{4}\int_{-1}^{1}(\mathrm{i}kx^2-2x)\frac{\mathrm{d}\hat\mu(x)}{\mathrm{d}x}\,\mathrm{d}x. \tag{21}$$

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[1] In solving this equation it is tacitly assumed that μ is continuous on $[-1,\infty)$ and is differentiable on $(-1,\infty)$. The first implication immediately follows the continuity of the potential; the second immediately follows the continuity of pressure. The continuity of the potential jump at the trailing edge, i.e., at 1, replaces the respective edge condition.

In (17),

$$\hat{L}_k\{\hat{\mu};x,z\} = \frac{1}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'}\frac{(x-x')\,\mathrm{d}x'}{(x-x')^2+z^2} - \frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{2\pi}\int_{1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx'}(x-x')\,\mathrm{d}x'}{(x-x')^2+z^2} \tag{22}$$

– it follows (10) by (16) and (18); whereas

$$\hat{G}_0(x) = -\mathrm{i}k\hat{f}(x) - \mathrm{d}\hat{f}(x)/\mathrm{d}x. \tag{23}$$

4. The setup

The only term in (17) that involves the distance of the wing from the ground is $\hat{L}_k\{\hat{\mu};x,2h\}$. It vanishes as h goes to infinity, thereby reducing the problem to that of solving

$$\hat{L}_k\{\hat{\mu};x,+0\} = \hat{G}_0(x) \quad \text{for each } x \in (-1,1), \tag{24}$$

subject to (19). This equation is well known in the theory of an oscillating wing section out of ground effect; it possesses a closed-form analytical solution cited, for example in Bisplinghoff et al. [1]; for completeness of this presentation it has been recapitulated in Appendix A, and pertinent formulae repeated in (42)–(44) below. At this point it suffices to say that its solution can be expressed as a combination of quadratures of its right-hand side and hence may be regarded as known. We shall mark this solution by the subscript '0'.

The ground-effect correction $\hat{\mu} - \hat{\mu}_0$ satisfies

$$\hat{L}_k\{\hat{\mu}-\hat{\mu}_0;x,+0\} = \hat{L}_k\{\hat{\mu};x,2h\} \quad \text{for each } x \in (-1,1), \tag{25}$$

which is the difference between (17) and (24). It vanishing as h goes to infinity suggests that Eq. (25), or, to the same end, Eq. (17), fits nicely the type of equation that can be solved asymptotically in a straightforward manner using $\varepsilon = 1/2h$, the ratio of the wing's quarter chord to its distance from the ground, as a small parameter. Yet all our attempts to obtain a *practical* solution of (17) using textbook-standard approach (as, for example, found in Van Dyke [6]) have failed. The reason for these failures has to do with the particular dependences of $\hat{L}_k\{\hat{\mu};x,+0\}$ and of $\hat{L}_k\{\hat{\mu};x,2h\} = \hat{L}_k\{\hat{\mu};x,1/\varepsilon\}$ on k and ε.

To demonstrate the problem, first consider $\hat{L}_k\{\hat{\mu};x,1/\varepsilon\}$ appearing on the right of (25) (or the left of (17)), in more details. It consists of two terms (see (22)): the one involving integration on $(-1,1)$,

$$\hat{W}\{\hat{\mu};x,1/\varepsilon\} = \frac{1}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'}\frac{\varepsilon^2(x-x')\,\mathrm{d}x'}{\varepsilon^2(x-x')^2+1} \tag{26}$$

– it represents the contribution of the image wing, and the other involving integration on $(1,\infty)$ – it represents the contribution of the entire image wake. Splitting the integration interval in this other term to $(1,x)$ and (x,∞), $\hat{L}_k\{\hat{\mu};x,1/\varepsilon\}$ can be recast as

$$\hat{L}_k\{\hat{\mu};x,1/\varepsilon\} = \hat{W}\{\hat{\mu};x,1/\varepsilon\} + \frac{\varepsilon^2}{2\pi\mathrm{i}k}\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k(1-x)}\big(\hat{J}\big(k/\varepsilon,\varepsilon(1-x)\big) - \hat{J}(k/\varepsilon,\infty)\big), \tag{27}$$

where $\hat{J}$ is a function on $(0,\infty)\times(0,2\varepsilon)$, such that

$$\hat{J}(\kappa,\zeta) = \kappa^2\int_0^{\zeta}\frac{\mathrm{e}^{-\mathrm{i}\kappa\xi}\xi\,\mathrm{d}\xi}{\xi^2+1} + 1. \tag{28}$$

The unity on the right-hand side of (28), which obviously cancels out in the parentheses of (27), was added to make $\hat{J}(\infty,\infty)$ zero.

Asymptotic solution of (17) will invariably involve representing $\hat{L}_k\{\hat{\mu};x,1/\varepsilon\}$ as a series in ε. With $\hat{L}_k\{\hat{\mu};x,1/\varepsilon\}$ given by (27), the expansion of its first two terms – those involving integration over finite intervals – presents no conceptual or algebraic difficulties; it yields

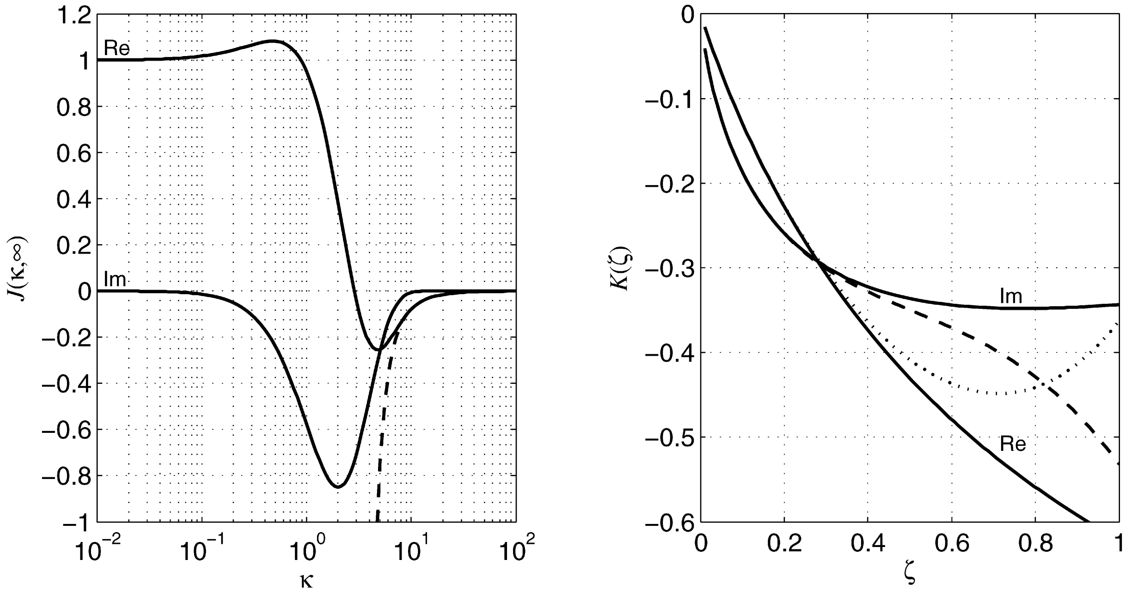

Fig. 1. Functions $\hat{J}(\cdot,\infty)$ and $\hat{K}$. Three-term expansion about infinity of the real part of $\hat{J}(\cdot,\infty)$ is marked by the dashed line on the left plate. Six-term expansion about zero of $\hat{K}$ is marked by the dotted and dashed lines on the rigth plate.

$$\hat{W}\{\hat{\mu};x,1/\varepsilon\} = \frac{\varepsilon^2}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'}(x-x')\,\mathrm{d}x' - \frac{\varepsilon^4}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'}(x-x')^3\,\mathrm{d}x' + \cdots, \tag{29}$$

$$\begin{aligned}\hat{J}\big(k/\varepsilon,\varepsilon(1-x)\big) &= \mathrm{e}^{-\mathrm{i}k(1-x)}\big(1+\mathrm{i}k(1-x)\big)\\ &\quad + \mathrm{e}^{-\mathrm{i}k(1-x)}\Big(2+3\big(1+\mathrm{i}k(1-x)\big)+\big(1+\mathrm{i}k(1-x)\big)^3-6\mathrm{e}^{\mathrm{i}k(1-x)}\Big)(\varepsilon/k)^2+\cdots.\end{aligned} \tag{30}$$

At the same time, expansion of the last term in (27), $\hat{J}(k/\varepsilon,\infty)$, although not difficult algebraically, is problematic conceptually. No power series about infinity (recall that $k/\varepsilon=\infty$ when $\varepsilon=0$) can approximate the behavior of $\hat{J}(\cdot,\infty)$,

$$\hat{J}(\kappa,\infty) = \kappa^2\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}\kappa\xi}\,\xi\,\mathrm{d}\xi}{\xi^2+1}+1 = \kappa\int_0^\infty\frac{\xi^2-1}{(\xi^2+1)^2}\sin(\kappa\xi)\,\mathrm{d}\xi + 1 - \mathrm{i}\frac{\pi}{2}\kappa^2\mathrm{e}^{-\kappa}, \tag{31}$$

in the vicinity of unity – see Fig. 1 on the left. In fact, three term series

$$\hat{J}(k/\varepsilon,\infty) = -6(k/\varepsilon)^{-2} - 120(k/\varepsilon)^{-4} - 5020(k/\varepsilon)^{-6}+\cdots \tag{32}$$

works fairly well for $k/\varepsilon > 7$ (say), but not lower. Since practical values of k/ε in applications can be anywhere between zero and infinity, the straightforward expansion of $\hat{L}_k\{\hat{\mu};x,1/\varepsilon\}$ with respect to ε appears inadequate *a priori*. By the way, a variant of this approach (employing $\tau=(\sqrt{1+4\varepsilon^2}-1)/2\varepsilon$ instead of ε) was used by Panchenkov [3] to solve the problem of an oscillating wing near a free surface; he gave no account on the applicability limits of his solution.

One may argue that if the problem stems from $\hat{J}(k/\varepsilon,\infty)$ being a function of ε, let $\kappa=k/\varepsilon$ serve as independent parameter, instead of k – see Darrozes [7]. This solves the problem of course, but creates a new one. With κ being now an independent parameter, $k=\kappa\varepsilon$ becomes a dependent one, and hence the frequency dependent part of

$$\hat{L}_k\{\hat{\mu};x,+0\} = \frac{1}{2\pi}\fint_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'}\frac{\mathrm{d}x'}{x-x'} - \frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{2\pi}\int_1^\infty\frac{\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x'}{x-x'} \tag{33a}$$

$$= \frac{1}{2\pi} \fint_{-1}^{1} \frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'} \frac{\mathrm{d}x'}{x - x'} - \frac{\hat{\mu}(1)}{2\pi} \frac{\hat{K}(\kappa\varepsilon(1-x))}{1-x}, \tag{33b}$$

becomes ε-dependent. Here, the bar across the integral sign indicates the principle value in Cauchy sense, whereas $\hat{K}$,

$$\hat{K}(\zeta) = \zeta \mathrm{e}^{\mathrm{i}\zeta}\big(\mathrm{si}(\zeta) + \mathrm{i}\,\mathrm{Ci}(\zeta)\big), \tag{34}$$

represents a combination of standard sine and cosine integrals (those with the infinite upper limit – see, for example, Jahnke and Emde [8]). Hence the use of κ as independent parameter necessitates an expansion of $\hat{K}(\varepsilon\kappa(1-x))$ into a series about zero. But even a six-terms series,

$$\hat{K}(\zeta) = \mathrm{i}\left(\mathrm{i}\frac{\pi}{2} + \gamma + \ln\zeta\right)\zeta - \left(\mathrm{i}\frac{\pi}{2} + \gamma + \ln\zeta - 1\right)\zeta^2 - \frac{\mathrm{i}}{2}\left(\mathrm{i}\frac{\pi}{2} + \gamma + \ln\zeta - \frac{3}{2}\right)\zeta^3 + \cdots \tag{35}$$

($\gamma = 0.5772\ldots$ being the Euler's constant) is still inadequate in approximating $\hat{K}(\zeta)$ for $\zeta > 0.4$ (say) – see Fig. 1 on the right. Since the argument of $\hat{K}$ in (33b) is, actually, $k(1-x)$, and since $1-x$ takes on values on $(0,2)$, being unable to approximate $\hat{K}$ for $k(1-x)$ above 0.4, implies keeping k below 0.2. This is a relatively modest limit for the reduced frequency, and from a practical point of view, a wider range is needed. Thus, if not inadequate *a priori*, this approach is at least ineffective.

5. Asymptotic solution of (17)

Returning to Eq. (17) let us temporary assume that its solution $\hat{\mu}(\cdot; k, \varepsilon)$ is known on $(-1, 1)$ for each $k \in (0, \infty)$ and $\varepsilon \in (0, \infty)$. What we have outlined (but did not finish) in the preceding section is the way to obtain asymptotic series for $\hat{\mu}(\cdot; k, \varepsilon)$ and $\hat{\mu}(\cdot; \kappa\varepsilon, \varepsilon)$ with respect to ε.

Now, since k and ε are just parameters in (17), we could have defined any combination of k and ε that is independent of x, which appears in (17), for example, k/ε or $\hat{J}(k/\varepsilon, \infty)$, as an additional independent parameter, say κ' or $\hat{J}''$, and obtained $\hat{\mu}'(\cdot; k, \varepsilon, \kappa')$ or $\hat{\mu}''(\cdot; k, \varepsilon, \hat{J}'')$ as its respective solution. Obviously, all solutions are equivalent, with $\hat{\mu}(\cdot; k, \varepsilon) = \hat{\mu}'(\cdot; k, \varepsilon, k/\varepsilon) = \hat{\mu}''(\cdot; k, \varepsilon, \hat{J}(k/\varepsilon, \infty))$ on $(-1, 1)$ for each $k \in (0, \infty)$ and $\varepsilon \in (0, \infty)$.

Taking this one step further, we suggest setting $\hat{J}(k/\varepsilon, \infty) = \hat{J}''$ in (27); solving (17) asymptotically with respect to ε(keeping, of course, $\hat{J}''$ independent of ε) so as to obtain an asymptotic series of $\hat{\mu}''(\cdot; k, \varepsilon, \hat{J}'')$ with respect to ε – and, after the solution has been obtained, resetting back $\hat{J}'' = \hat{J}(k/\varepsilon, \infty)$. Yielding $\hat{\mu}''(\cdot; k, \varepsilon, \hat{J}(k/\varepsilon, \infty)) = \hat{\mu}(\cdot; k, \varepsilon)$ as its infinite sum, the resulting series solution is as valid as any of the two standard solutions attempted at the beginning of the preceding section. Only this time it can be obtained with neither conceptual nor algebraic difficulties.

In fact, with both k and $\hat{J}''$ independent of ε, Eq. (17) involves only powers of ε – see (27), (29) and (30). Accordingly, we can assume (subject to *a posteriori* verification) that its solution is given by a simple power series,

$$\hat{\mu} = \hat{\mu}_0 + \sum_{j=1} \varepsilon^{2j} \hat{\mu}_j, \tag{36}$$

and, concurrently,

$$\hat{c}_z = \hat{c}_{z,0} + \sum_{j=1} \varepsilon^{2j} \hat{c}_{z,j}, \qquad \hat{c}_m = \hat{c}_{m,0} + \sum_{j=1} \varepsilon^{2j} \hat{c}_{m,j}. \tag{37}$$

It immediately follows by direct substitution that $\hat{c}_{z,j}$ and $\hat{c}_{m,j}$ are given by variants of (20) and (21) with '$\hat{\mu}_j$' instead of '$\hat{\mu}$'. It also follows that $\hat{\mu}_j$ satisfies

$$\hat{L}_k\{\hat{\mu}_j; x, +0\} = \hat{G}_j(x) \quad \text{for each } x \in (-1, 1), \tag{38}$$

subject to

$$\hat{\mu}_j(-1) = 0; \tag{39}$$

the first equation follows (17) by (27), (29), (30) and (36); the last equation follows (19) by (36). In (38), $\hat{G}_0$ has been already defined in (23), whereas

$$\hat{G}_1(x) = \frac{1}{2\pi \mathrm{i}k}\big(\hat{c}_{z,0} - \hat{\mu}_0(1)\mathrm{e}^{\mathrm{i}k(1-x)}\hat{J}''\big); \tag{40}$$

$$\hat{G}_2(x) = \frac{1}{2\pi \mathrm{i}k}\big(\hat{c}_{z,1} - \hat{\mu}_1(1)\mathrm{e}^{\mathrm{i}k(1-x)}\hat{J}''\big) - \frac{1}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}_0(x')}{\mathrm{d}x'}(x-x')^3\,\mathrm{d}x' + \frac{\hat{\mu}_0(1)}{2\pi \mathrm{i}k^3}\big(2 + 3\big(1+\mathrm{i}k(1-x)\big) + \big(1+\mathrm{i}k(1-x)\big)^3 - 6\mathrm{e}^{\mathrm{i}k(1-x)}\big); \tag{41}$$

etc.

Eq. (38) has the same form as (24), and hence possesses a closed-form analytical solution expressing $\hat{\mu}_j$ (and, consequently, $\hat{c}_{z,j}$ and $\hat{c}_{m,j}$) as quadratures of $\hat{G}_j$. Relevant formulae can be found in (A.9), (A.10), (A.20) and (A.21) of Appendix A; upon adjusting the notation, they give

$$\hat{\mu}_j(1) = 2\mathrm{e}^{-\mathrm{i}k}B_0(k)\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}_j(\zeta)\,\mathrm{d}\zeta, \tag{42}$$

$$\hat{c}_{z,j} = 2\mathrm{i}k\int_{-1}^{1}\sqrt{1-\zeta^2}\hat{G}_j(\zeta)\,\mathrm{d}\zeta + 2C(k)\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}_j(\zeta)\,\mathrm{d}\zeta, \tag{43}$$

$$\hat{c}_{m,j} = \int_{-1}^{1}\sqrt{1-\zeta^2}\left(1-\frac{\mathrm{i}k\zeta}{2}\right)\hat{G}_j(\zeta)\,\mathrm{d}\zeta + \frac{C(k)-1}{2}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}_j(\zeta)\,\mathrm{d}\zeta. \tag{44}$$

In (42)–(44), B_0 and C are combinations

$$B_0(k) = -\frac{2}{\mathrm{i}k\pi(H_1^{(2)}(k) + \mathrm{i}H_0^{(2)}(k))}, \tag{45}$$

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + \mathrm{i}H_0^{(2)}(k)}, \tag{46}$$

of Hankel functions. Two additional combinations of Bessel functions,

$$B_1(k) = B_0(k)\big(J_0(k) - \mathrm{i}J_1(k)\big), \tag{47}$$

$$B_2(k) = \mathrm{i}J_1(k) + C(k)\big(J_0(k) - \mathrm{i}J_1(k)\big), \tag{48}$$

will appear in (49)–(51) below. All B's and C are bounded by unity over $(0,\infty)$; all equal unity at zero; all the B's eventually vanish at infinity, whereas C tends to $1/2$.

Generic quadratures associated with $\hat{G}_1$ and $\hat{G}_2$ have been addressed in Appendix B. Upon identifying the pertinent constants in (B.1)–(B.4) from (40) and (41) one should find no difficulty to verify that

$$\hat{\mu}_1(1) = \frac{1}{\mathrm{i}k}\big(\hat{c}_{z,0}\mathrm{e}^{-\mathrm{i}k}B_0(k) - \hat{\mu}_0(1)B_1(k)\hat{J}''\big), \tag{49}$$

$$\hat{c}_{z,1} = \hat{c}_{z,0}\frac{\mathrm{i}k + 2C(k)}{2\mathrm{i}k} - \hat{\mu}_0(1)\frac{\mathrm{e}^{\mathrm{i}k}}{\mathrm{i}k}B_2(k)\hat{J}'', \tag{50}$$

$$\hat{c}_{m,1} = \frac{\hat{c}_{z,1}}{4} - \frac{\hat{c}_{z,0}}{8} = \hat{c}_{z,0}\frac{C(k)}{4\mathrm{i}k} - \hat{\mu}_0(1)\frac{\mathrm{e}^{\mathrm{i}k}}{4\mathrm{i}k}B_2(k)\hat{J}''. \tag{51}$$

ε^4-order terms ($j=2$) follow the same formulae, but they become too lengthy to be presented here (and probably elsewhere) in explicit form. Collecting it all together, i.e., collecting (36)–(37) and (49)–(51), and resetting $\hat{J}'' = \hat{J}(k/\varepsilon,\infty)$, yields

$$\hat{\mu}(1) = \hat{\mu}_0(1) + \frac{\varepsilon^2}{\mathrm{i}k}\big(\hat{c}_{z,0}\mathrm{e}^{-\mathrm{i}k}B_0(k) - \hat{\mu}_0(1)B_1(k)\hat{J}(k/\varepsilon,\infty)\big) + \mathrm{O}(\varepsilon^4), \tag{52}$$

$$\hat{c}_z = \hat{c}_{z,0} + \frac{\varepsilon^2}{2\mathrm{i}k}\big(\hat{c}_{z,0}\big(\mathrm{i}k + 2C(k)\big) - 2\hat{\mu}_0(1)\mathrm{e}^{\mathrm{i}k}B_2(k)\hat{J}(k/\varepsilon,\infty)\big) + \mathrm{O}(\varepsilon^4), \tag{53}$$

$$\hat{c}_m = \hat{c}_{m,0} + \varepsilon^2\hat{c}_{z,0}\frac{C(k)}{4\mathrm{i}k} - \varepsilon^2\hat{\mu}_0(1)\frac{\mathrm{e}^{\mathrm{i}k}}{4\mathrm{i}k}B_2(k)\hat{J}(k/\varepsilon,\infty) + \mathrm{O}(\varepsilon^4), \tag{54}$$

where $\hat{\mu}_0$, $\hat{c}_{z,0}$ and $\hat{c}_{m,0}$ are given by the respective variants of (42)–(44).

After $\hat{J}''$ has been reset as $\hat{J}(k/\varepsilon,\infty)$, solution (52)–(54) can be manipulated at will. For example, expanding $\hat{\mu}(1)$, $\hat{c}_z$ and $\hat{c}_m$ into Maclaurin series with respect to ε (k kept constant) yields the respective first terms

$$\hat{\mu}(1) = \hat{\mu}_0(1) + \frac{\varepsilon^2}{\mathrm{i}k}\hat{c}_{z,0}\mathrm{e}^{-\mathrm{i}k}B_0(k) + \mathrm{O}(\varepsilon^4), \tag{55}$$

$$\hat{c}_z = \hat{c}_{z,0} + \frac{\varepsilon^2}{2\mathrm{i}k}\hat{c}_{z,0}\big(\mathrm{i}k + 2C(k)\big) + \mathrm{O}(\varepsilon^4), \tag{56}$$

$$\hat{c}_m = \hat{c}_{m,0} + \varepsilon^2\hat{c}_{z,0}\frac{C(k)}{4\mathrm{i}k} + \mathrm{O}(\varepsilon^4), \tag{57}$$

in the asymptotic solution attempted at the beginning of Section 4. Similarly, taking k to zero yields the respective first two terms

$$c_z = \mu(1) = -2\delta\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}\,\mathrm{d}\zeta - \delta\varepsilon^2\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}(3-2\zeta)\,\mathrm{d}\zeta + \mathrm{O}(\delta\varepsilon^4), \tag{58}$$

$$c_m = -\delta\left(1+\frac{\varepsilon^2}{2}\right)\int_{-1}^{1}\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}\sqrt{1-\zeta^2}\,\mathrm{d}\zeta + \mathrm{O}(\delta\varepsilon^4), \tag{59}$$

in the asymptotic solution for a wing in quasi-steady motion in ground effect. In deriving (58) and (59) we have used (42)–(44) and (23) to obtain amplitudes, and (16) to obtain the loads. For flat airfoil at angle of attack, and for parabolic-arc airfoil, Eq. (58) accords the results cited in Eqs. (7.74a) and (7.74b) of Katz and Plotkin [9].

6. Heave oscillations

The above formulae should hold for any mode of wing oscillations, provided, of course, that it complies with the set of assumptions stated at the beginning. Still, heave oscillations deserve, we believe, a special attention, due to both the (relative) availability of data to compare with, and peculiar behavior at low oscillation frequencies that may elucidate the limitations of the present theory.

For the case of heave oscillations, the first three terms in the respective series for the lift and pitching moment coefficients can be found in Eqs. (C.5) and (C.6) of Appendix C. The available numerical data consists of DLM simulations of Nitta [10] for symmetrical airfoil oscillating parallel with the ground one-quarter chord above it, and Euler simulations of Moryossef and Levi [11] (to be referred to as ‘ML’) addressing cambered airfoil oscillating with relatively large amplitude ($\delta = 0.2$) about few positions situated 0.35 to 0.7 chords from the ground. For the sake of discussion, Nitta’s results[2] for the lift ($\varepsilon = 1$), taken from her Fig. 9, and 0.7-chords-from-the-ground results[3] of ML ($\varepsilon = 0.36$), taken from their Figs. 13 and 14a, have been recompiled in Fig. 2.

Although not designed to work at $\varepsilon = 1$, present results fit fairly well the numerical simulations of Nitta, and fit nicely the numerical simulations of ML, but only for k that is grater than, say, 0.2. As k goes to zero, ML results imply that the lift oscillations gradually become in phase with the wing position and do not vanish, whereas present results,

[2] In Nitta’s article, the moment is referred to 0.3-chord-from-leading-edge point, its unit is half of that used in (C.6), and the positive wing displacement is reversed. Nitta’s results appearing in Fig. 2 of this exposition have been adjusted to fit the present notation.

[3] To fit the present notation, the amplitude of the lift oscillations appearing in Fig. 14a of ML has been divided by the amplitude of the heave oscillations.

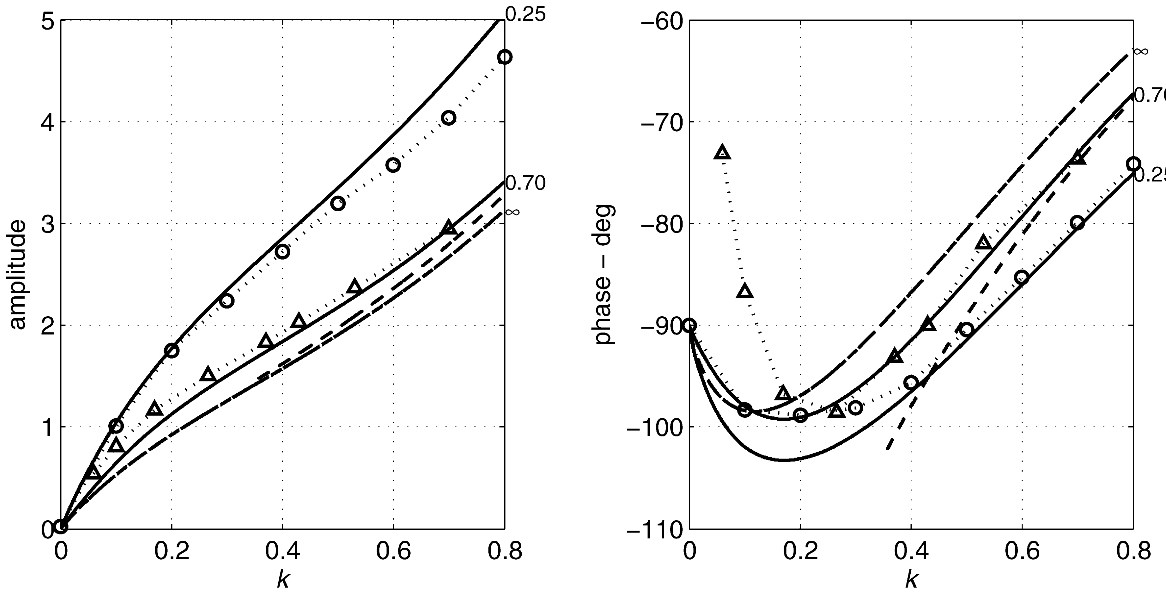

Fig. 2. Amplitude and phase of $\hat{c}_z$ for heave oscillations. Order ε^2 expansion (53) is represented by the solid lines with the distance from the ground (in chords) appearing to the right of each line. Order ε^2 expansion (56) at 0.7 chords from the ground is represented by the dashed line. Simulations of Nitta [10] at quarter chord above the ground, and those of Moryossef and Levi [11] at 0.7-chords-from-the-ground are represented by circles and triangles respectively.

and those of Nitta, suggest that the lift should lag 90 degrees behind the wing position and vanish in the limit. Indeed, Eq. (C.5) implies that in the leading order with respect to k,

$$\hat{c}_z = -2\pi \mathrm{i} k(1+\varepsilon^2) + \cdots. \tag{60}$$

The reason for this disagreement is related with the large steady-state constituent of the lift produced by the cambered airfoil of ML's study. In fact, the lift of a non-oscillating cambered wing in ground effect (58) is a function of the wing's distance from the ground. When the wing heaves, even at vanishingly slow rate, it samples various distances from the ground, and hence changes its lift coefficient. Since the magnitude of these changes is independent of oscillation frequency, at sufficiently low frequency – where the right-hand side of (60) vanishes – the lift becomes in phase with the wing position. But this is a δ^2-order effect (wing's lift is of order δ and it varies with the distance to the ground, which, in turn, changes by δ); it is beyond the scope of the present theory, which is consistent only at the order δ, but not δ^2.

Order ε^2 expansion (56) for $\varepsilon = 0.36$ (0.7 chords from the ground) is shown by dashed lines in Fig. 2. As could have been expected from the discussion of Section 4, it is clearly inadequate, in particular at low frequencies. The only difference between (56) and (53) is in the '$\hat{J}$-term', which has been associated earlier (Section 4) with the effect of the image wake. The absence of this term in (56) – its reminiscent can be found in the higher-order terms – is equivalent with the neglect of the image wake effect. Indeed, the image wake influence on the wing vanishes when the period $2\pi/k$ of alternating vortices comprising the wake becomes small as compared with their distance $2h = 1/\varepsilon$ to the wing; these are the exact circumstances in the limit where ε goes to zero and k kept constant. But k and ε in Fig. 2 are comparable quantities, and hence the effect of the image wake cannot be ignored. In fact, the agreement between (53) and the Nitta's results at $\varepsilon = 1$ suggests that this effect is probably the dominant one.

7. Summary

The solution for oscillating wing section in weak ground effect was constructed in two steps. In the first one, the problem was linearized with respect to δ, the ratio of the wing transversal displacement to its semi-chord. This step was essential in separating time-dependent constituents of the aerodynamic loads from time-independent ones, and, ultimately, in reducing the problem to that of an oscillating infinitesimally thin section.

In the second step, the integral equation governing the problem of an oscillating infinitesimally thin section in ground effect was solved asymptotically using ε, the ratio of the wing quarter-chord to the average distance between the wing trailing edge and the ground, as a small parameter. The main complication here had to do with the structure of the integral equation. It involved four groups of terms: (i) those depending on k only, (ii) those depending on ε only, (iii) those depending on the ratio $\kappa = k/\varepsilon$ of the two, and (iv) those independent of the two. Taking ε to zero with k independent required the terms in group (iii) to be expanded about $\kappa = \infty$. With practical values of κ lying beyond the convergence radius of this expansion, this approach failed. Taking ε to zero with κ independent required the terms in group (i) to be expanded about $k = 0$. Although not *a priori* impossible, this approach required too many terms in the expansion to become practical. The ultimate solution was to treat κ (or, rather, $J(\kappa, \infty)$) as a third, independent, parameter, that comes in addition to ε and k; and restore its dependence on the two only after the solution has been obtained. This solution recovered well known analytical results when $k \to 0$, and, with only three terms, fitted nicely the available numerical simulations.

Formally, the three-terms solution is asymptotically accurate to within terms of the order ε^4, but without obtaining all terms in the series (or, what is probably simpler, comparing it with numerous numerical simulations) it is impossible to specify its practical applicability limit. One may hope that this limit should correspond to the values of ε for which ε^4 is less than a tolerable approximation error, say 0.01. Should this be the case, the three term solution may become practical for the values of ε as high as about one-third – i.e., for the wing flying as low as three-quarters chord above the ground. Good agreement with numerical simulations we have obtained at the distances of 0.7 and 0.25 chords above the ground seems to support (but certainly not prove) this conjecture. If this is indeed the case, the present theory should be applicable for aircraft wings during ground roll and low flight, and even for racing cars high set front wings.

Appendix A

Given an operator $\hat{L}_k\{\cdot; x, +0\}$, such that

$$\hat{L}_k\{\hat{\mu}; x, +0\} = \frac{1}{2\pi} \int_{-1}^{1} \frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'} \frac{\mathrm{d}x'}{x - x'} - \frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{2\pi} \int_{1}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x'}{x - x'}, \tag{A.1}$$

and given a function $\hat{G}$, piecewise continuous on $(-1, 1)$, we seek a function $\hat{\mu}$, finite on $(-1, 1)$, which satisfies

$$\hat{L}_k\{\hat{\mu}; x, +0\} = \hat{G}(x) \quad \text{for each } x \in (-1, 1), \tag{A.2}$$

vanishes at -1, and which derivative is singular at -1, and finite at 1. Solution technique is after Schwarz [12] as described in Bisplinghoff et al. [1]. Since we refer to different stages of the solution in the manuscript, we take the liberty to recapitulate it briefly in the following lines, emphasizing the most conspicuous results.

Consider first an auxiliary problem. Given a function g, piecewise continuous on $(-1, 1)$, we seek a function f, integrable on $(-1, 1)$, that satisfies

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{f(x')\,\mathrm{d}x'}{x - x'} = g(x) \quad \text{for each } x \in (-1, 1) \tag{A.3}$$

has singularity at -1, and is finite at 1. Solution of this problem is commonly known as Söhngen inversion; it yields

$$f(x) = -\frac{2}{\pi} \sqrt{\frac{1 - x}{1 + x}} \int_{-1}^{1} \sqrt{\frac{1 + x'}{1 - x'}} \frac{g(x')\,\mathrm{d}x'}{x - x'}. \tag{A.4}$$

A proof of this statement can be found in several references – the most accessible of them is, perhaps, in Ashley and Landahl [5]; the proof is on pages 91–93 thereat.

Now, rewriting (A.2) as

$$\frac{1}{2\pi}\fint_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x')}{\mathrm{d}x'}\frac{\mathrm{d}x'}{x-x'}=\hat{G}(x)+\frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{2\pi}\int_{1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x'}{x-x'}\quad\text{for each } x\in(-1,1);\tag{A.5}$$

one will find its right-hand side satisfying the requirements imposed on the function g in (A.3); accordingly, the pertinent solution of (A.5) is

$$\frac{\mathrm{d}\hat{\mu}(x)}{\mathrm{d}x}=-\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\fint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{\hat{G}(\zeta)\,\mathrm{d}\zeta}{x-\zeta}-\frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{\pi^2}\sqrt{\frac{1-x}{1+x}}\fint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{\mathrm{d}\zeta}{x-\zeta}\int_{1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x'}{\zeta-x'}.\tag{A.6}$$

It can be simplified by changing the order of integration in the second term, and integrating with respect to ζ; the result is

$$\frac{\mathrm{d}\hat{\mu}(x)}{\mathrm{d}x}=-\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\fint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{\hat{G}(\zeta)\,\mathrm{d}\zeta}{x-\zeta}+\frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{\pi}\sqrt{\frac{1-x}{1+x}}\int_{1}^{\infty}\sqrt{\frac{x'+1}{x'-1}}\frac{\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x'}{x-x'}.\tag{A.7}$$

Since $\hat{\mu}(-1)=0$ by definition, integration respect to x between -1 and 1 on both sides of (A.7) yields a simple equation,

$$\hat{\mu}(1)=2\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}(\zeta)\,\mathrm{d}\zeta-\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}\int_{1}^{\infty}\left(\sqrt{\frac{x'+1}{x'-1}}-1\right)\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x',\tag{A.8}$$

involving $\hat{\mu}(1)$ only. Assuming $\mathrm{Im}\,k=-0$, the integral on the right can be separated into two finite terms: one of them (involving the '-1') cancels out with $\hat{\mu}(1)$ on the left-hand side; the other one (involving the square-root) yields a pair of Hankel functions. The final result can be put into the form

$$\hat{\mu}(1)=-\frac{4}{\mathrm{i}k\pi\mathrm{e}^{\mathrm{i}k}(H_1^{(2)}(k)+\mathrm{i}H_0^{(2)}(k))}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}(\zeta)\,\mathrm{d}\zeta.\tag{A.9}$$

Once $\hat{\mu}(1)$ has been found, $\hat{\mu}$ follows direct integration of (A.7) between -1 and x. With substitution $\eta=\sqrt{(1-x)/(1+x)}$ the integration is fairly straightforward, albeit lengthy; using (A.8) its result can be recast as

$$\hat{\mu}(x)=\hat{\mu}(1)+\frac{2}{\pi}\int_{-1}^{1}\Lambda_1(x,\zeta)\hat{G}(\zeta)\,\mathrm{d}\zeta-\frac{2\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{\pi}\int_{1}^{\infty}\Lambda_2(x,x')\mathrm{e}^{-\mathrm{i}kx'}\mathrm{d}x',\tag{A.10}$$

where Λ_1 and Λ_2 are a pair of auxiliary functions, respectively defined on $(-1,1)\times(-1,1)$ and $(-1,1)\times(1,\infty)$, such that

$$\Lambda_1(x,\zeta)=\ln\left|\frac{\sqrt{(1-x)(1+\zeta)}+\sqrt{(1+x)(1-\zeta)}}{\sqrt{(1-x)(1+\zeta)}-\sqrt{(1+x)(1-\zeta)}}\right|,\tag{A.11}$$

$$\Lambda_2(x,\xi)=\tan^{-1}\sqrt{\frac{(\xi+1)(1-x)}{(\xi-1)(1+x)}}.\tag{A.12}$$

Given $\hat{\mu}$ and its derivative on $(-1,1)$, the pressure jump $\hat{p}$ is given by an appropriate variant of (7), whereas the lift and the pitching moment about mid-chord point are given by (20) and (21); in particular

$$\hat{c}_z=-\frac{1}{2}\int_{-1}^{1}\hat{p}(x)\,\mathrm{d}x=\int_{-1}^{1}\left(\mathrm{i}k\hat{\mu}(x)+\frac{\mathrm{d}\hat{\mu}(x)}{\mathrm{d}x}\right)\mathrm{d}x=(1+\mathrm{i}k)\hat{\mu}(1)-\mathrm{i}k\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x)}{\mathrm{d}x}x\,\mathrm{d}x,\tag{A.13}$$

$$\hat{c}_m=\frac{1}{4}\int_{-1}^{1}\hat{p}(x)x\,\mathrm{d}x=-\frac{\mathrm{i}k}{4}\hat{\mu}(1)+\frac{\mathrm{i}k}{4}\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x)}{\mathrm{d}x}x^2\,\mathrm{d}x-\frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}\hat{\mu}(x)}{\mathrm{d}x}x\,\mathrm{d}x.\tag{A.14}$$

Substituting (A.7) in the last two equations, changing the order of integration, and carrying out the integration with respect to x yields

$$\hat{c}_z = (1+\mathrm{i}k)\hat{\mu}(1) + 2\mathrm{i}k\int_{-1}^{1}\sqrt{1-\zeta^2}\hat{G}(\zeta)\,\mathrm{d}\zeta - (\mathrm{i}k)^2\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}\int_{1}^{\infty}\big(x' - \sqrt{x'^2-1}\big)\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x', \tag{A.15}$$

$$\begin{aligned}\hat{c}_m = {} & -\frac{\mathrm{i}k}{4}\hat{\mu}(1) + \frac{\mathrm{i}k}{4}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}(\zeta)\,\mathrm{d}\zeta - \frac{(\mathrm{i}k)^2\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{8}\int_{1}^{\infty}\sqrt{\frac{x'+1}{x'-1}}\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x' \\ & - \frac{\mathrm{i}k}{2}\int_{-1}^{1}\sqrt{1-\zeta^2}\hat{G}(\zeta)\zeta\,\mathrm{d}\zeta + \frac{(\mathrm{i}k)^2\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{4}\int_{1}^{\infty}\big(x'^2 - x'\sqrt{x'^2-1}\big)\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x' \\ & + \int_{-1}^{1}\sqrt{1-\zeta^2}\hat{G}(\zeta)\,\mathrm{d}\zeta - \frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{2}\int_{1}^{\infty}\big(x' - \sqrt{x'^2-1}\big)\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x'.\end{aligned} \tag{A.16}$$

The sum of the second and the third terms in (A.16) is zero by (A.8) under the assumption that $\operatorname{Im} k = -0$; after integration by parts in the third term in (A.15) and in the fifth term in (A.16) these become

$$\hat{c}_z = 2\mathrm{i}k\int_{-1}^{1}\sqrt{1-\zeta^2}\hat{G}(\zeta)\,\mathrm{d}\zeta - \frac{\pi\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{2}H_1^{(2)}(k), \tag{A.17}$$

$$\begin{aligned}\hat{c}_m &= \int_{-1}^{1}\sqrt{1-\zeta^2}\left(1-\frac{\mathrm{i}k\zeta}{2}\right)\hat{G}(\zeta)\,\mathrm{d}\zeta - \frac{\mathrm{i}k\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{4}\int_{1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}x'}{\sqrt{x'^2-1}} \\ &= \int_{-1}^{1}\sqrt{1-\zeta^2}\left(1-\frac{\mathrm{i}k\zeta}{2}\right)\hat{G}(\zeta)\,\mathrm{d}\zeta + \frac{(\mathrm{i}k)(\mathrm{i}\pi)\hat{\mu}(1)\mathrm{e}^{\mathrm{i}k}}{8}H_0^{(2)}(k).\end{aligned} \tag{A.18}$$

Final expressions for the aerodynamic loads now follow these by (A.9); with

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + \mathrm{i}H_0^{(2)}(k)}, \tag{A.19}$$

they can be recast as

$$\hat{c}_z = 2\mathrm{i}k\int_{-1}^{1}\sqrt{1-\zeta^2}\hat{G}(\zeta)\,\mathrm{d}\zeta + 2C(k)\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}(\zeta)\,\mathrm{d}\zeta, \tag{A.20}$$

$$\hat{c}_m = \int_{-1}^{1}\sqrt{1-\zeta^2}\left(1-\frac{\mathrm{i}k\zeta}{2}\right)\hat{G}(\zeta)\,\mathrm{d}\zeta + \frac{C(k)-1}{2}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\hat{G}(\zeta)\,\mathrm{d}\zeta. \tag{A.21}$$

Appendix B

Pertinent particular cases of (A.9), (A.20) and (A.21) are associated with $\hat{G}$ having either polynomial or exponential form. Accordingly, let

$$\hat{G}(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + b_0\mathrm{e}^{\mathrm{i}k(1-x)}, \tag{B.1}$$

where a_0, a_1, a_2, a_3 and b_0 are, at this stage, arbitrary constants. Upon evaluating the associated integrals, explicit expressions for $\hat{\mu}(1)$, $\hat{c}_z$ and $\hat{c}_m$ can be recast as

$$\hat{\mu}(1) = \frac{\pi}{4}\mathrm{e}^{-\mathrm{i}k} B_0(k)(8a_0 + 4a_1 + 4a_2 + 3a_3) + 2\pi B_1(k)b_0, \tag{B.2}$$

$$\hat{c}_z = \frac{\pi}{4}\mathrm{i}k(4a_0 + a_2) + \frac{\pi}{4}C(k)(8a_0 + 4a_1 + 4a_2 + 3a_3) + 2\pi \mathrm{e}^{\mathrm{i}k} B_2(k)b_0, \tag{B.3}$$

$$\hat{c}_m = -\frac{\pi}{32}\big(2(4+\mathrm{i}k)a_1 + 4a_2 + (6+\mathrm{i}k)a_3\big) + \frac{\pi}{16}C(k)(8a_0 + 4a_1 + 4a_2 + 3a_3) + \frac{\pi}{2}\mathrm{e}^{\mathrm{i}k} B_2(k)b_0, \tag{B.4}$$

where C, B_0, B_1 and B_2 denote particular combinations of Bessel functions. C has been defined in (A.19), whereas

$$B_0(k) = -\frac{2}{\mathrm{i}k\pi(H_1^{(2)}(k) + \mathrm{i}H_0^{(2)}(k))}, \tag{B.5}$$

$$B_1(k) = -\frac{2(J_0(k) - \mathrm{i}J_1(k))}{\mathrm{i}k\pi(H_1^{(2)}(k) + \mathrm{i}H_0^{(2)}(k))}, \tag{B.6}$$

$$B_2(k) = \mathrm{i}J_1(k) + C(k)\big(J_0(k) - \mathrm{i}J_1(k)\big) = \mathrm{i}\frac{J_1(k)Y_0(k) - Y_1(k)J_0(k)}{H_1^{(2)}(k) + \mathrm{i}H_0^{(2)}(k)}. \tag{B.7}$$

All the B's equal unity at zero, zero at infinity, and their absolute values are bounded by unity over $(0, \infty)$.

Appendix C

For heave oscillations, $\hat{G}_0 = -\mathrm{i}k$. In notation of (B.1) it implies $a_0 = -\mathrm{i}k$ with all other constants being identically zero. Hence,

$$\hat{\mu}_0(1) = -2\pi \mathrm{i}k\mathrm{e}^{-\mathrm{i}k} B_0(k), \tag{C.1}$$

$$\hat{c}_{z,0} = -\pi \mathrm{i}k\big(\mathrm{i}k + 2C(k)\big), \tag{C.2}$$

$$\hat{c}_{m,0} = -\frac{\pi}{2}\mathrm{i}kC(k), \tag{C.3}$$

by (42)–(44) and (B.2)–(B.4). The expressions for the lift and moment coefficients follow substitution of these in (49)–(51), and back in (36) and (37), and, in general, further back in (16). Skipping the last substitution, the results are

$$\hat{\mu}(1) = -2\pi \mathrm{i}k\mathrm{e}^{-\mathrm{i}k} B_0(k) - \varepsilon^2\pi \mathrm{e}^{-\mathrm{i}k} B_0(k)\big(\mathrm{i}k + 2C(k) - 2\hat{J}(k/\varepsilon, \infty)B_1(k)\big) + \cdots, \tag{C.4}$$

$$\hat{c}_z = -\pi \mathrm{i}k\big(\mathrm{i}k + 2C(k)\big) - \varepsilon^2\frac{\pi}{2}\big(\big(\mathrm{i}k + 2C(k)\big)^2 - 4\hat{J}(k/\varepsilon, \infty)B_0(k)B_2(k)\big) + \cdots, \tag{C.5}$$

$$\hat{c}_m = -\frac{\pi}{2}\mathrm{i}kC(k) - \varepsilon^2\frac{\pi}{4}\big(\big(\mathrm{i}k + 2C(k)\big)C(k) - 2\hat{J}(k/\varepsilon, \infty)B_0(k)B_2(k)\big) + \cdots. \tag{C.6}$$

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